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CONTROLLING THE NONLINEAR DYNAMICAL SYSTEM OF A LOTKA VOLTERRA SYSTEM USING A RECURSIVE BACKSTEPPING METHOD

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Abstract: A control strategy for a class of biological population model simulating prey-predator relationship using backstepping scheme is presented, the increasing rate of predator population has the effect of reducing the growth rate of the preys, and this reduction depends on the number of encounters between individuals of the two species. The Lotka Volterra model controller is systematically designed via a recursive procedure that skillfully interlaces the choice of a Lyapunov function with the control to eliminate the undesirable chaotic oscillation. Theoretically, it has been proved that the error signal can exponentially converge to zero. Numerical simulations are presented to show the effectiveness and feasibility of the method of control. The simulations revealed that by choosing the proper parameters of the controller, there will be a stabilize chaotic dynamics of the system to the stable equilibrium point thereby making the output signal to track all kinds of reference signals.

Keywords: Chaos control, backstepping, model, numerical simulation, signal

Introduction

The Lotka-Volterra model is a nonlinear model that allows for mathematical calculation (Baosheng *et al.*, 2011), it has been used extensively for modeling evolutionary game dynamics (Madeo & Mocenni, 2015), economic theory (Huu & Costa-lima, 2014), biological systems (Křivan & Priyadarshi, 2015), chemical reactions (Magyar, Szederkényi, & Hangos, 2008), plasma physics (Laval & Pellat, 1975). Since Ott *et al.* (1990) first successfully controlled chaotic system using the OGY method and the subsequent realization of the idea of master-slave synchronization of chaotic systems and its possible application to secure communications by Pecora & Caroll (1990). Recently, many researchers are looking into the dynamics of population with time delays and this can be applied in the control of mankind, animals and environmental population (Tayir *et al.*, 2016). The field of chaos control and tracking has triggered an avalanche of publications both from the theorists and the control engineers due to its ubiquitous applications especially in areas such as information science, medicine, biology and engineering. A formal statement of a control problem typically begins with a model of the system to be controlled (controlled plant) and a model of the control objective (control goal). A typical control goal systems is to transform a chaotic trajectory into a periodic one, control theory can be referred to as stabilization of an unstable or equilibrium system. From the viewpoint of system structure, most of the proposed chaos control methods focus on the study of self-synchronization for low dimensional chaotic systems (Blekhman *et al.*, 1997; Yang *et al.*, 2019).

A specific feature of this problem is the possibility of achieving the goal by means of a virtual control action. In many cases, synchronization and chaotization can also be achieved by small control (Andrievskii & Fradkov, 2003) several classes of models are considered in paper reviews on chaos control and most common class is described in state space by differential equations. In many control applications, to design a controller that can alter or modify the behavior and response of an unknown plant to meet certain performance requirements can be tedious and challenging. This process is characterized by a certain number of inputs 'u' and outputs 'y'. The plant inputs (u) are processed to produce several plant outputs y that represents the output response of the plant. The task of the designed control is to choose the input 'u' so that the response y(t) satisfies certain given performance requirements. The three pioneering and most actively developing major branches of research such as feed forward control (also called non-feedback or open-loop (vibrational)) control based on periodic system excitation (Morgul, 1999), OGY method (based on linearization of Poincaré map) (Ott *et al.*, 1990) and time-delayed feedback also referred to as Pyragas method (Pyragas, 1992) are actively explored. Nevertheless, they face numerous unsolved problems concerned mostly with a justification of chaos control methods. Control of chaos is closely related to nonlinear control and many well-developed engineering methods of control such as linear and nonlinear control; adaptive control; neural networks; fuzzy control has been shown to be applicable and efficient for chaotic systems (Vincent, & Yu, 1991). The existence of a feedback passifying, that is, making passive, the closed-loop system is the prerequisite for efficiency (attainment of the objective) of the majority of the above approaches. From the point of view of "small control," the methods based on passivity offer an advantage because they allow one to attain the goal independently of the gain.

In nonlinear control theory, the problem of stabilizing the invariant objective manifold h(x) = 0 by a small control is solved by Tian (1999) using the method of macro variables proposed by Kolesnikov (1987). Other methods of the modern theory of nonlinear control such as the theory of center manifold (Friedel *et al.*, 1997) the backstepping procedure and the methods of iterative sign (Mascolo & Grassi, 1999), the method of passivity-based design (Miroshnik & Nikiforov, 2000), the method of variable-structure systems (VS-system) (Fang *et al.*, 2000) the theory of absolute stability (Suykens *et al.*, 1998), the H∞-optimal design (Curran *et al.*, 1997; Suykens *et al.*, 1997) and a combination of the direct Lyapunov method and linearization by feedback (Loria *et al.*, 1998), were used to solve the problems of stabilization about the given state or the objective manifold.

It was reported by Xu *et al.* (2011) that, some researchers investigated the equilibrium of prey predator system and they were able to study the properties of Hopf bifurcation for the system by using normal form theory. However, despite numerous publications, control of chaos remains an area of intensive research and only a few strict facts were established there, and many issues remain open. In view of the wide scope of possible applications, this area is of interest both to the theorists and the control engineers. The present work aims to help gain an insight into the state-of-art in this vast domain of research and its most interesting applications.

Materials and Methods

Backstepping design

The backstepping design is used in this paper and it can be define as a systematic Lyapunov function control technique which can be applied to pure-feedback systems, block strict-feedback systems and recursive feedback systems (Krstic *et al.*, 1995) but this work focused on recursive backstepping systems. The backstepping approach provides recursive method for stabilizing the origin of a system in strict-feedback form. Considering a system of the form:

$$\begin{cases} \dot{X} = f_x(X) + g_x(X)z_1 \\ \dot{z}_1 = f_1(X, z_1) + g_1(X, z_1)z_2 \\ \vdots \\ \dot{z}_i = f_i(X, z_1, z_2, \ldots, z_{i-1}, z_i) + g_i(X, z_1, z_2, \ldots, z_{i-1}, z_i)z_{i+1} \\ \vdots \\ \dot{z}_{k-1} = f_{k-1}(X, z_1, z_2, \ldots, z_{k-1}) + g_{k-1}(X, z_1, z_2, \ldots, z_{k-1})z_k \end{cases}$$

$$for 1 \leq i < k-1$$

Where

1. $X \in R^n with\, n \geq 1$
2. $Z_1, Z_2, \ldots, Z_i, \ldots, Z_{k-1}, Z_k$ are scalars,
3. u is a scalar input to the system,
4. $f_x, f_1, f_2, \ldots, f_i, \ldots, f_{k-1}, f_k$ vanish at the origin (i.e., f_i (0,0, …, 0) = 0),
5. $g_1, g_2, \ldots, g_i, \ldots, g_{k-1}, g_k$ are non zero over the domain of interest (i.e., $g_i(X, Z_{1,\ldots}, Z_K)$ $\neq 0$ for $1 \leq i \leq k$).

Also assume that the subsystem $\dot{X} = f_x(X) + g_x$(x) u_x(X)

Is stabilized to the origin (i.e., X = 0) by some known control u_x(X) such that u_x(0) = 0. The back stepping-designed control input u has its most immediate stabilizing impact on state z_n.

The state z_n then acts like a stabilizing control on the state Z_{n-1} before it.

6. This process continues so that each state z_i is stabilized by the fictitious "control" $Z_i + 1$.
7. The Backstepping approach determine how to stabilized the **x** subsystem using z_1, and then determine how to make the next state z_2 drive z_1 to the control required to stabilize X. Therefore, the process steps backward from X out of the strict-feedback form system until the ultimate control u is designed.

Lotka-Volterra model

The model was proposed by an Italian mathematician, Umberto Volterra in 1926 (Volterra, 1926). He used a mathematical model of a predator-prey situation to explain why Italian fishermen caught a larger percentage of sharks and other predator fish (corresponding decrease in prey fish) during the first World War 1 in the Adriatic Sea than was true both before and after the war. At the same time in the United States, Volterra equations were derived independently by Alfred Lokta (1920) to describe a hypothetical chemical reaction in which the chemical concentrations oscillate. The simplest model of predator-prey interactions is the Lotka-Volterra model.

Let x(t) denote the population of the prey, and let y(t) denote the population of the predators. In the absence of the predators, the prey population would have a birth rate greater than its death rate, and consequently would grow according to the exponential model of population growth, i.e. the growth rate of the population would be proportional to the population itself. The presence of the predator population has the effect of reducing the growth rate, and this reduction depends on the number of encounters between individuals of the two species. Since it is reasonable to assume that the number of such encounters is proportional to the number of individuals of each population, the reduction in the growth rate is also proportional to the product of the two populations, i.e.,

$$\dot{x} = ax - \alpha xy \quad (1)$$

More so, the predator population is a dependent population, it depends on the prey population for its food supply and it's natural to assume that in the absence of the prey population, the predator population would actually decrease, i.e. the growth rate would be negative. Furthermore, the (negative) growth rate is proportional to the population. The presence of the prey population would provide a source of food, so it would increase the growth rate of the predator species. By the same reasoning used for the prey species, this increase would be proportional to the product of the two populations, i.e.

$$\dot{y} = -cy + \gamma xy. \quad (2)$$

The Lotka-Volterra model in its original form is given as

$$\begin{aligned} \dot{x} &= ax - \alpha xy \\ \dot{y} &= -cy + \gamma xy \end{aligned} \quad (3)$$

$\dot{x}$ and $\dot{y}$ represent the instantaneous growth rates of the two populations; t represents time; it is known that the variation of the parameters (a, α, c, γ) results in various types of dynamical behavior including quasi-periodicity and transition to chaos.

Backstepping design for the controller

Here, we introduce $u(t)$ into Eq. 3 in order to design a nonlinear controller that eliminates undesirable chaotic oscillation. The aim is to choose an appropriate Lyapunov function V whose time derivative $\dot{V}$ is made non-positive,

i.e., $\dot{V} <= 0$ by properly choosing the differential equation of the adaptive law.
Thus, we have the strict feedback form as;

$$\dot{x} = ax - \alpha xy$$
$$\dot{y} = -cy + \gamma xy + u(t) \quad (4)$$

By redefining the variables

$$\text{If} \quad \dot{x} = \dot{x}_1, \quad \dot{y} = \dot{x}_2 \quad (5)$$

Therefore;

$$\dot{x}_1 = ax_1 - \alpha x_1 x_2$$
$$\dot{x}_2 = -cx_2 + \gamma x_1 x_2 + u(t) \quad (6)$$

In the recursive backstepping scheme, it is required that the state space x_1 and x_2 are to take desired values $x_1 d$ and $x_2 d$. Then the error signals are

$$e_1 = x_1 - x_1 d$$
$$e_2 = x_2 - x_2 d \quad (7)$$

Differentiating the error signals in Eq. 7 where $x_1 d$ is an equilibrium point of the system and $x_2 d = c_1 e_1$; c_1 being a constant. Using Eq. 6 into Eq. 7 to obtain

$$\dot{e}_1 = ae_1 - \alpha e_1(e_2 + c_1 e_1)$$
$$\dot{e}_2 = \gamma(e_1\{e_2 + c_1 e_1\}) - c(e_2 + c_1 e_1) - c_1[ae_1 - \alpha e_1(e_2 + c_1 e_1)] + u(t) \quad (8)$$

Introducing the Lyapunov function

$$V = \frac{1}{2}\sum_{i=1}^{2} k_1 e_1^2 \quad (9)$$

The time derivative of equation (9) is

$$\dot{V} = \sum_{i=1}^{2} k_i e_i \dot{e}_i = k_1 e_1 \dot{e}_1 + k_2 e_2 \dot{e}_2 \quad (10)$$

Substituting Eq. 8 into Eq. 10 and setting $k_1 = 0$, $k_2 = 1, c_1 = 1, \dot{v} = 0$ and $k_2 = 1$. Since

$$\dot{V} = k_2 e_2 \dot{e}_2$$

Hence

$$\dot{V} = -ce_2 + \gamma e_1 e_2 - cc_1 e_1 + \gamma c_1 e_1 e_1 - ac_1 e_1 + \alpha c_1 c_1 e_1 e_1$$
$$\dot{V} = e_2\{(\gamma e_1 + \alpha e_1 - c)(e_2 + e_1)\} - ae_1 + u(t) \quad (11)$$

By making the control $u(t)$ the subject of formula

$$u(t) = (c - \gamma e_1 - \alpha e_1)(e_2 + e_1) + ae_1 \quad (12)$$

Then $\dot{V} = -ce_2^2 < 0$ is negative definite. Thus, the control goal is achieved and according to Lasalle-Yoshizawa theorem, it follows that all the solutions of system in Eq. 6 converge to equilibrium. Thus, the control goal is achieved and will present the numerical simulations to verify the effectiveness.

Results and Discussion

The temporal evolution of the model was investigated by solving numerically Eq. 3 using the controller obtained in Eq. 12. For this purpose, the Runge-Kutta scheme was used. In the calculation, a step size of 0.0001 was used. The initial conditions are $(x_o, y_o) = (0.1, 0.01)$ with computation over the interval [0, 120]. The constant parameters are given by $a = 0.4, c = 0.3$ and $\gamma = 0.05$.

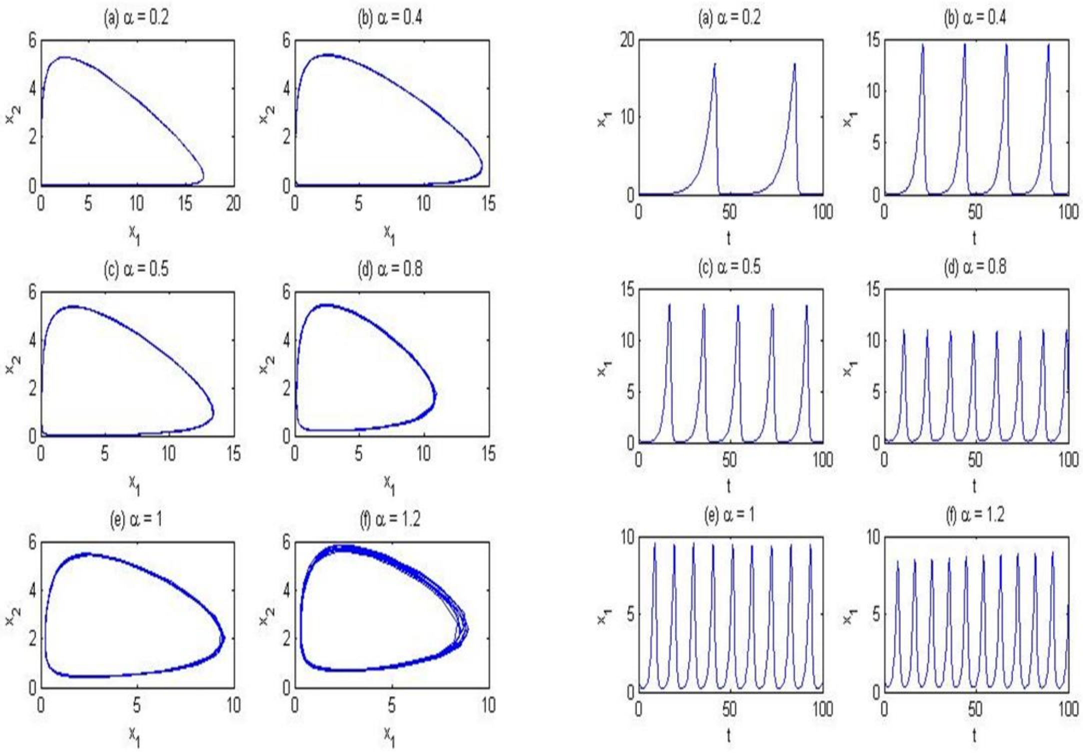

Fig. 1: Phase plot and Time Series of the system showing quasi-periodicity for varying values of α with other parameter values $a = 0.4, c = 0.3$ and $\gamma = 0.05$. $\alpha = 0.2, 0.4, 0.6, 0.8, 1.0, 1.2$ for Fig(s). 1a-1f, respectively

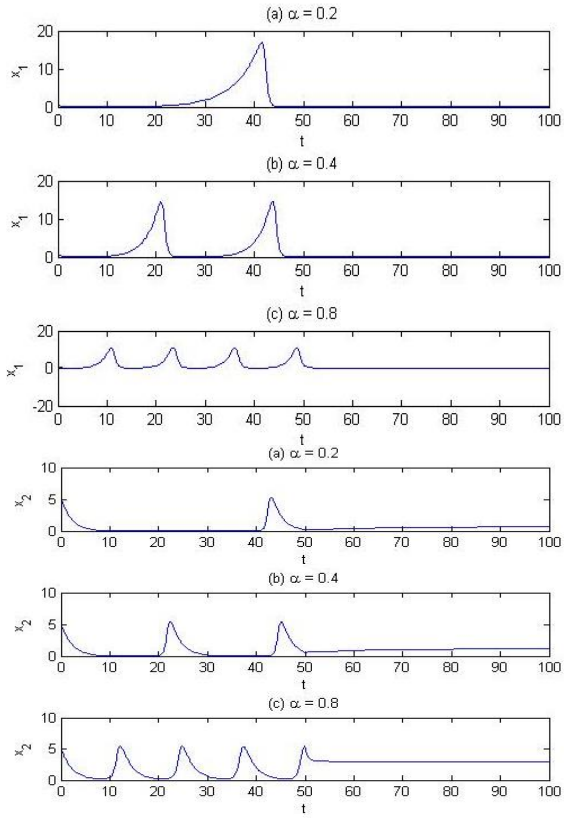

Fig. 2: Time series of the system showing its dynamical behaviour is stable after the controller is triggered at t=50 for $\alpha = 0.2, 0.4$ and 0.8 for Fig. 2a, 2b, and 2c, respectively. The system other parameters are $a = 0.4, c = 0.3$ and $\gamma = 0.05$.

In Fig. 2, the controller u(t) is activated at t =50. As expected, one can observe that the output trajectory is asymptotically driven to a stable state as soon as the controller is applied, preventing it from entering the quasi-periodic state. This validates the effectiveness of the proposed control method.

Conclusion

Conclusively, the relationship between the predators and the preys was an exclusive irregular pattern. The research article was used to introduce a backstepping control method that can be used to solve the chaotic dynamics of the system based upon the Lyapunov stability theory. A controller was introduced into the differential equations for the simulations by choosing the proper parameters. The numerical simulation showed the dynamics of the model and the stabilizing of the chaotic dynamics of the model to attain a stable equilibrium point.

Conflict of Interest

Authors declare there is no conflict of interest in this study.

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